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Physica A 355 (2005) 602–618
PHYSICA A
www.elsevier.com/locate/physa

Congestion schemes and Nash equilibrium in complex networks

Juan A. Almendral[a], Luis López[b], Vicent Cholvi[c], Miguel A.F. Sanjuán[a,*]

[a]*Grupo de Dinámica No Lineal y Teoría del Caos, Departamento de Matemáticas y Física Aplicadas y Ciencias de la Naturaleza, Universidad Rey Juan Carlos, Móstoles, 28933 Madrid, Spain*
[b]*Laboratorio de Algoritmia Distribuida y Redes, Departamento de Informática, Estadística y Telemática, Universidad Rey Juan Carlos, Móstoles, 28933 Madrid, Spain*
[c]*Departamento de Lenguajes y Sistemas Informáticos, Universitat Jaume I, 12071 Castellón, Spain*

Received 21 December 2004
Available online 11 May 2005

Abstract

Whenever a common resource is scarce, a set of rules are needed to share it in a fairly way. However, most control schemes assume that users will behave in a cooperative way, without taking care of guaranteeing that they will not act in a selfish manner. Then, a fundamental issue is to evaluate the impact of cheating. From the point of view of game theory, a Nash equilibrium implies that nobody can take advantage by unilaterally deviating from this stable state, even in the presence of selfish users. In this paper we prove that any efficient Nash equilibrium strongly depends on the number of users, if the control scheme policy does not record their previous behavior. Since this is a common pattern in real situations, this implies that the system would be always out of equilibrium. Consequently, this result proves that, in practice, oblivious control schemes must be improved to cope with selfish users.

PACS: 89.75.Fb; 89.65.Gh; 89.70. + c

Keywords: Complex networks; Game theory; Nash equilibrium

*Corresponding author. Tel.: +34 916647443; fax: +34 916647490.
E-mail address: msanjuan@escet.urjc.es (M.A.F. Sanjuán).

doi:10.1016/j.physa.2005.03.023

1. Introduction

The problem of control schemes constitutes a largely studied issue in the past few years [1–7]. Many systems nowadays are based on the principle of sharing a common resource, e.g., a communication link, among different users. Consequently one of the main objectives of such schemes is to establish a number of rules guaranteeing that the common resources are shared in a fair way among users.

However, most of these schemes require users to behave in a cooperative way, so that they respect some "social responsible" rules. Moreover, without forcing end users to adopt a centralized mandated policy controlling their behavior, it is not possible to guarantee that they will not act in a selfish manner. Then, it seems a main issue to evaluate the impact of having users acting this way.

An example that illustrates the above-mentioned scenario is the control scheme used by the TCP/IP protocol, which is currently the dominant protocol in the internet. By using it, users control the injection rate of packets into the communication network by means of a pair of parameters. When users detect that the network is overloaded, which is done by means of some control messages, they decrease their injection rate by a half, thus alleviating the network's load. However, the adherence to this scheme is voluntary in nature, and some users may decide to act in a selfish manner and not to decrease its injection rate. As it has been evaluated by several authors [8,9], this may lead to a congestion collapse that only benefits selfish users. Therefore, it is interesting to know how cooperative users may "fight back" against unsupportive ones.

Another example of congestion can be found in social networks, mainly when they are based on the traditional hierarchical paradigm. Despite the problems of congestion showed by this topology, large companies prefer this hierarchical organization because it is the only way to keep their activities under a strict control. But with the growth of the companies, the number of specialized activities grows also, and it is needed to introduce a non-hierarchical communication to maintain the efficiency. However, nobody knows how to control a non-hierarchical organization. The hope is that some global "self-organizing" order emerges by means of a horizontal protocol [10]. Consequently, it is important to analyze the features which such a successful protocol must have.

Game theory constitutes a good mathematical tool for analyzing the interaction of decision makers with conflicting interests [11,12]. From a game-theoretic perspective, users are considered the *game players* and congestion control schemes establish the *game rules*.

We regard *players* as agents that issue requests for a common resource selfishly (i.e., they are only concerned about their own good). Hence, the utility function of each player, which is the parameter to be maximized, is assumed to be equal to the number of requests that have been served per unit time.

The *rules* of the game are determined by the management policy of the common resource. Here, we consider policies that are *oblivious*, i.e., that do not differentiate between requests belonging to different agents, and that have a limited storage capacity for pending requests. Moreover, requests issued after such a limit is

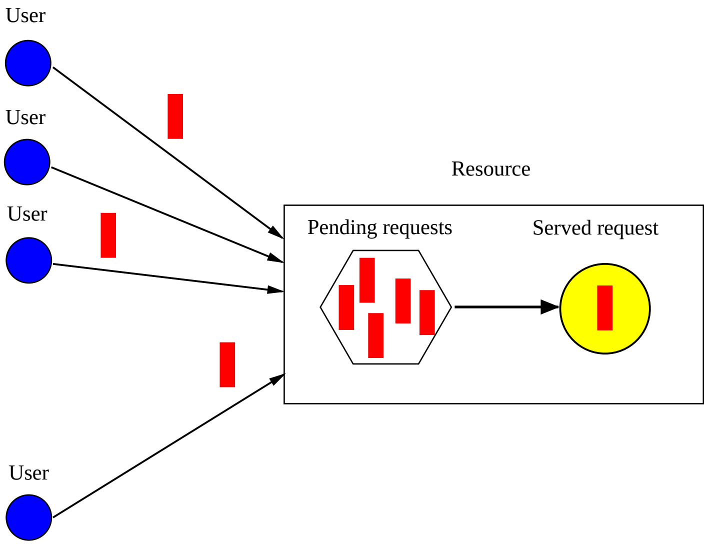

Fig. 1. Scenario considered in this paper. The requests sent to use the common resource are represented here with filled rectangles. These requests are stored up to a certain limit until they are chosen obliviously to be served, one at each time.

reached are simply discarded. Fig. 1 shows an illustration of the above-mentioned scenario.

Once the players and the rules have been fixed, the next step is the definition of a utility function. But the problem here is that its concrete form depends on the assumptions made for the network under analysis.

For example, if we consider N players in a communication network, the request rate of the ith player (also referred to as *load*) can be modelled by a Poisson process with average rate λ_i and the utility function (also known as the *goodput*) can be written as $\mu_i = \lambda_i(1 - p(\lambda))$ [13], where $p(\lambda)$ is the discarding probability due to an average aggregate load of $\lambda \equiv \sum_{i=1}^{N} \lambda_i$ and an average service time of unity.[1]

Although the utility function is different for many other games, in the rest of the paper the goodput given in the former example to ease the reading is used. Nonetheless, we will show that the results derived from this particular goodput can be generalized to a broad set of problems.

An important concept in game theory is the *Nash equilibrium*. In our context, a Nash equilibrium is a scenario where no selfish user has a reason to unilaterally deviate from its current state, because he is acting in an optimal way. Clearly, being in a Nash equilibrium means that we are in a stable state in the presence of selfish users.

[1]Note that there is no loss of generality if it is assumed that the service rate of the system is normalized to 1.

In a Nash equilibrium, no player can increase his goodput by either increasing or decreasing their request rate λ_i. Then, the following condition must be satisfied:

$$\left.\frac{\partial \mu_i}{\partial \lambda_i}\right|_{\lambda^*} = 0, \quad i = 1, \ldots, N ,$$

where λ^* is the average aggregate request rate at equilibrium. If it is assumed that $p(\lambda)$ is a differentiable function, the former condition can be rewritten as

$$q(\lambda^*) + \lambda_i^* q'(\lambda^*) = 0 ,$$

where $q(\lambda) \equiv 1 - p(\lambda)$ for simplicity.

Since we are interested in a *symmetric equilibrium*, which imposes $\lambda_i^* = \lambda^*/N$, the Nash condition becomes

$$q(\lambda^*) + \frac{\lambda^*}{N} q'(\lambda^*) = 0 . \tag{1}$$

It is interesting to note that this symmetry condition implies that the goodput at equilibrium is the same for all players, which is the only way to guarantee that the obtained policy is fair.

On the other hand, given a solution for the Nash condition, it is also desirable that such a solution has a good efficiency. A solution is *efficient* when the aggregate goodput at equilibrium μ^*, which is defined as

$$\mu^* \equiv \sum_{i=1}^{N} \mu_i^* = \sum_{i=1}^{N} \lambda_i^* q(\lambda^*) = \lambda^* q(\lambda^*) ,$$

verifies that $\lim_{N\to\infty} \mu^*$ is a positive constant, otherwise it means that no requests are being processed.

Observing Eq. (1) we may remark that λ^* is, in general, a function of the number N. Hence, the load of any of the players at equilibrium λ_i^* also depends on N. In this situation, it is interesting to define a parameter measuring the increase on λ_i^* when N changes. With this purpose, we use the *sensitivity coefficient* $\Delta_i(N)$, which can be defined as [14]

$$\Delta_i(N) \equiv \lambda_i^*(N) - \lambda_i^*(N-1) .$$

Note that $\Delta_i(N)$ is a measurement of how difficult it is for player i to reach a new equilibrium when the number of users increases from $N-1$ to N. For practical purposes, it will be interesting to obtain oblivious policies having no sensitivity to N, that is, $\Delta_i(N) = 0$. We say that a policy is *reachable* in a practical situation if it has no sensitivity to N. This would guarantee that, once all players have reached the equilibrium, they will be able to maintain it without the need of passing a transient period of time searching their new Nash conditions.

In this paper, we show that, with the above-presented assumptions, i.e., greedy users using an oblivious control scheme policy, being in a symmetric efficient Nash equilibrium will be highly sensitive. Hence, from a practical point of view, if N changes rapidly, which occurs in very realistic situations, it means that the system would be always out of equilibrium.

2. Impossibility of efficient solutions to the Nash condition

2.1. The Nash condition in the continuum limit

As it has been stated previously, the average aggregate load at equilibrium λ^* derived from the Nash condition depends on N, the number of agents involved in the network. Hence, λ^* is a discrete function $\lambda^* : \mathbb{N} \to \mathbb{R}^+$, which for every value of N returns the λ^* imposed by the Nash condition for N agents.

However, although λ^* is a discrete function of N, it is always possible to regard λ^* as a twice differentiable function $f : [1, \infty) \to \mathbb{R}^+$ such that $f(N) = \lambda^*(N)$ for all positive integer N. Note that the function λ^* can be seen geometrically as a set of points in the plane located at $(N, \lambda^*(N))$, where N is an integer. Then, the definition of f simply reflects the fact that we can always draw a curve (twice differentiable, for technical reasons) passing through them.

Therefore, when this continuum limit is taken, Eq. (1) can be seen as the following condition, which holds for all $v \geqslant 1$:

$$q[f(v)] + \frac{f(v)}{v} q'[f(v)] = 0 ,$$

where, according to the definition of f, the derivative must be understood as $()' \equiv \mathrm{d}/\mathrm{d}f()$.

Consequently, if the notation $\dot{()} \equiv \mathrm{d}/\mathrm{d}v()$ and $q(v) \equiv q[f(v)]$ is used for simplicity, the Nash condition in the continuum limit is written as the following first-order ordinary differential equation:

$$q(v) + \frac{f(v)}{v} \frac{\dot{q}(v)}{\dot{f}(v)} = 0 ,$$

whose solution can be written formally as

$$q(v) = D\mathrm{e}^{-I(v)} , \tag{2}$$

where D is a constant of integration and $I(v)$ is defined in this manner

$$I(v) \equiv \int \frac{\dot{f}(v)}{f(v)} v \, \mathrm{d}v .$$

Although Eq. (2) is a formal solution to the Nash condition, it is enough to demonstrate that any efficient solution must tend asymptotically to a positive constant (see Appendix A). Namely, $f(v)$ must verify

$$0 < \lim_{v \to \infty} f(v) < \infty .$$

This implies that $f(v)$ can always be written as

$$f(v) \equiv f_\infty [1 + \tilde{f}(v)] , \tag{3}$$

with f_∞ a positive constant and $\tilde{f}(v)$ a twice differentiable function verifying $\lim_{v \to \infty} \tilde{f}(v) = 0$ and $\tilde{f}(v) > -1$, for all $v \geqslant 1$.

2.2. Efficient solutions to the Nash condition

Since not all $f(v)$ verifying Eq. (3) is an efficient solution, we need more conditions to fix the set of efficient solutions. There are two results which are useful for this purpose.

The first one is a sufficient condition (see Appendix B) which states that if $\tilde{f}(v)$ is a twice differentiable function behaving asymptotically as

$$\dot{\tilde{f}}(v)v^2 \sim \frac{1}{v^\alpha} \quad \text{with } \alpha > 0 ,$$

then, an efficient solution is derived.

The second result is a necessary condition (see Appendix C) which states that if the Nash condition is verified efficiently, the following equation holds:

$$\lim_{v \to \infty} \dot{\tilde{f}}(v)v^2 = 0 .$$

Notice that because both conditions are similar but not equal, nothing can be said about some functions. The following function is an example:

$$\tilde{f}(v) = \frac{-1}{v \ln v} \Longrightarrow \dot{\tilde{f}}(v)v^2 = \frac{1 + \ln v}{(\ln v)^2} \sim \frac{1}{\ln v} .$$

Nevertheless, the set of efficient solutions are functions such that asymptotically tend to a constant faster than $1/v$. Then, in terms of the load, this result tells us that, at equilibrium, any efficient solution must have a λ^* which falls with N to a constant value faster than $1/N$.

Then, as our equilibrium is assumed to be symmetric, the load of any of the players at equilibrium changes asymptotically in the form $\lambda_i^*(N) \sim 1/N$. Hence, the sensitivity coefficient for any player behaves like $\Delta_i(N) \sim (1/N) - [1/(N-1)] \sim 1/N^2$ in that limit.

This allows us to conclude that in situations where the number of players changes rapidly, which occurs often, the efficient equilibrium of any oblivious efficient policy is not easily reachable, because the load of players depend strongly on the number of current players.

At this point, we would like to remark that it is possible to have situations with an efficient Nash equilibrium. What we proved here is that, in order to remain in an efficient equilibrium, users must adapt their request rate whenever the number of users change. Furthermore, such an adaption will be very significant and, consequently, it will likely require some time to be realized.

Therefore, if the number of users changes rapidly, the system will be always evolving from one equilibrium to another, without reaching any of them (or being in an equilibrium only during a short time interval). That is, the system would be always out of equilibrium.

2.3. Generality of the results

The former results have been deduced from a particular utility function, when the policy in use is oblivious and the equilibrium is supposed symmetric. However, in this context, the key feature of any utility function is that there is only one parameter to maximize it. For example, in the goodput used throughout the paper this parameter was the average aggregate load.

In this section we prove that our conclusions can be applied to many other functions depending on one parameter. At this point it is important to remark that our results are applicable if the model depends on several parameters, but it is understood that they are fixed in the utility function.

The results we have derived with the goodput μ_i are based on the following two assumptions:

$$\left.\frac{\partial \mu_i}{\partial \lambda_i}\right|_{\lambda^*} = 0 \quad \text{and} \quad \mu^* = \sum_{i=1}^{N} \mu_i^* > 0 \,,$$

which represent the Nash condition to obtain an equilibrium and the restriction imposed by the efficiency, respectively. Recall that $\mu_i = \lambda_i q(\lambda)$ and the only features required to derive the results were that q was a differentiable probability.

Hence, if we can prove that other utility function v_i verifies these conditions if and only if the goodput μ_i does, the results obtained for μ_i can also be applied to v_i and vice versa.

For example, if we consider the family of utility functions $v_i = \kappa\lambda^N q(\lambda)$, with κ a positive constant and $q(\lambda)$ a function in the range [0, 1] for all positive λ, it is easy to prove that

$$\left.\frac{\partial v_i}{\partial \lambda_i}\right|_{\lambda^*} = \left.\left(\kappa N \lambda^{N-1} \frac{\partial \mu_i}{\partial \lambda_i}\right)\right|_{\lambda^*} = 0 \iff \left.\frac{\partial \mu_i}{\partial \lambda_i}\right|_{\lambda^*} = 0 \,,$$

$$v^* = \sum_{i=1}^{N} v_i^* = \left.(\kappa \lambda^{N-1} \mu)\right|_{\lambda^*} > 0 \iff \mu^* > 0 \,,$$

thus we can apply to $v_i = \kappa\lambda^N q(\lambda)$ the results obtained in this paper.

Likewise, if g is a strictly increasing differentiable function such that $g(0) = 0$, the function $\xi_i = g(v_i)$ satisfies the following equations:

$$\left.\frac{\partial \xi_i}{\partial \lambda_i}\right|_{\lambda^*} = g'(v_i^*) \left.\frac{\partial v_i}{\partial \lambda_i}\right|_{\lambda^*} = 0 \iff \left.\frac{\partial v_i}{\partial \lambda_i}\right|_{\lambda^*} = 0 \,,$$

$$\xi^* = N g\left(\frac{v^*}{N}\right) > 0 \iff v^* > 0 \,,$$

which implies that our conclusions are also verified by any utility function with the form $g(v_i)$.

In particular, if we fix $g(x) = x^{\beta/N}$ being $\beta > 0$, the conclusions derived for the utility function μ_i hold for $v_i^{\beta/N}$. But $\tilde{\kappa} \equiv \kappa^{\beta/N}$ is a positive constant and

$\tilde{q}(\lambda) \equiv q(\lambda)^{\beta/N}$ is a differentiable function in $[0,1]$, thus this paper is applicable to the set of functions $\tilde{\kappa}\lambda^{\beta}\tilde{q}(\lambda)$ with $\tilde{\kappa}$, $\beta > 0$ and $\tilde{q}(\lambda)$ a differentiable probability function.

On the other hand, given a utility function $u(\lambda)$, it is reasonable to suppose that it is non-negative with $u(0) = 0$, otherwise we may redefine the utility function to satisfy this condition. Likewise, it can be assumed that $u(\lambda)$ is an increasing function in some neighborhood of $\lambda = 0$, i.e., any increase of λ in this neighborhood means some increase in the utility.

Then, the former assumptions imply that if $u(\lambda)$ cannot be written as $\lambda^{\beta}u^{*}(\lambda)$, with $\beta > 0$ and u^{*} a function such that $u^{*}(0)$ is a positive constant, we can always find a transformation g such that $g(u(\lambda)) = \lambda^{\beta}u^{*}(\lambda)$. For example, $u(\lambda) = \ln(1+\lambda)$ grows slower than any power of λ, but the function $g(x) = e^{x} - 1$ makes that $g(u(\lambda)) = \lambda$.

In other words, no matter how slow $u(\lambda)$ grows in the neighborhood of $\lambda = 0$, we can "inflate" it up to some power of λ by means of a function g. This can be done, for example, applying $g(x) = e^{x} - 1$ several times (if needed). Notice that, for all integer n, the composition $g^{n} \equiv g \circ g \circ \overset{(n)}{\cdots} \circ g$ is also a strictly increasing differentiable function such that $g^{n}(0) = 0$.

Although we can find a function g_1 such that $g_1(u(\lambda)) = \lambda^{\beta}u_1^{*}(\lambda)$, it is not guaranteed that u_1^{*} is a function in the range $[0,1]$. However, if we use the transformation $g_2(x) = \tanh x$, the resulting function takes the following form:

$$g_2(g_1(u(\lambda))) = \tanh\left(\lambda^{\beta}u_1^{*}(\lambda)\right) = \lambda^{\beta}u_2^{*}(\lambda)\,,$$

where u_2^{*} is now a function in the range $[0,1]$. Note that, due to the definition of g_1 and g_2, the composition $g \equiv g_1 \circ g_2$ is a strictly increasing differentiable function such that $g(0) = 0$.

Therefore, we can assure the applicability of our conclusions to a broad family of functions. We have shown that if the utility function $u(\lambda)$ is non-negative, with $u(0) = 0$ and increasing in some neighborhood of $\lambda = 0$, we can find a function g such that $g(u(\lambda)) = \lambda^{\beta}u^{*}(\lambda)$, with $\beta > 0$ and u^{*} a differentiable function defined in the range $[0,1]$.

Then, since $g(x)$ is a strictly increasing differentiable function with $g(0) = 0$, the conclusions of this paper can be applied to $g(u(\lambda))$ if and only if they can be applied to $u(\lambda)$. But we showed that this paper can be applied to the set of utility functions $\lambda^{\beta}u^{*}(\lambda)$, thus also to u, which is a rather general function.

3. Real-world congestion schemes

In this section, we analyze two real-world congestion schemes and show how their behavior can be greatly affected by the result presented in this paper.

3.1. The TCP/IP protocol

As it has been pointed out in the introduction of this paper, the TCP/IP protocol is the dominant protocol in the Internet. Furthermore, most of the current scheduling policies in Internet are oblivious, e.g., first in first out (FIFO). Therefore, taking into account that the adherence to the TCP/IP control scheme is voluntary in nature, our result shows that, in the presence of selfish users, TCP/IP control schemes cannot impose a Nash equilibrium.

To assess the importance of the above-mentioned effect in real situation, we note that there are already protocols, patented and owned by commercial companies [15], that "exploit" the "weakness" of TCP/IP protocol by means of behaving in a selfish manner. Such protocols offer goodputs (i.e., number of requests served per unit time) that are higher than those offered by the TCP/IP protocol, but at a cost of penalizing the performance of the latter.

Unfortunately, our result implies that it is not possible to obtain an internet router capable of providing an equilibrium without having to maintain a record of the requests performed by each user.

A conclusion somehow similar to ours, when focussed in communication systems, has been reported by other authors. For example, Dutta et al. [14] assume that the discarding probability is a non-decreasing and convex function. Furthermore, they assumed that their sensitivity coefficient depends on the number of flows in an exponential fashion.

Such assumptions, although simplify the proofs, are arbitrary. On the contrary, our result is completely general. Surprisingly, we noted that their assumption about the sensitivity coefficient constitutes a sufficient condition to obtain an efficient solution.

Altman et al. presents in Ref. [16] a detailed analysis when the utility function of each user is taken as the ratio of some positive power of the total throughput of that user to the average delay seen by the user. And, in Ref. [17], they consider a routing problem in networks defined by a directed graph with a polynomial cost function.

However, when all these utility functions are studied for a symmetric equilibrium, they are functions of only one parameter and we can apply the results derived in this paper.

3.2. Social networks

Several papers [5] have appeared in the literature reporting that social networks and computer networks share similar features in terms of information transfer. Some recent work [18] remarks that the problem of congestion often arises in social networks, mainly when these are based on the traditional hierarchical paradigms used by companies and organizations.

An illustrative example is telecommunication or digital cable/satellite TV companies. Usually, installers must communicate with a central site in order to validate new services, e.g., a home installation. At the central site, requests are queued and managed in the receiving order, but only up to a certain threshold, above it, the installer is required to communicate later on (i.e., the request is dropped).

Clearly, the number of requests may have peaks at given moments. If installers act in a cooperative way, they will communicate again after some time. But some of them may act unfairly and communicate immediately, trying to reduce their waiting time, at a cost of increasing the waiting time of the other employees.

The only way to attain a reachable and efficient equilibrium, being fair at the same time (i.e., giving equal opportunities to any requester) is to track their activities over time measuring their individual degrees of cooperation, and acting accordingly to these measurements. Namely, any strategy taking into account only the current state and not the individual past will fail.

4. Conclusions

In this paper, the effect of having users that act in a selfish manner when using a common resource has been thoroughly analyzed. We have shown that, if the policy used to manage the common resource is oblivious (i.e., if it does not differentiate between requests belonging to different users) then any efficient Nash equilibrium will highly depend on the number of users, in the sense that they must adapt their request rate in a significant manner. Taking into account that, in many realistic situations, the number of users changes rapidly and that the time needed to adapt from one equilibrium to another one can be significant, this means that the system will be most of the time out of equilibrium.

As illustrative examples, we point out a pair of congestion schemes where the above-mentioned effect may have a real impact.

Acknowledgements

This work has been partially supported by the Spanish Ministry of Science and Technology under project number BFM2003-03081.

Appendix A

Lemma A.1. *If $f(v)$ is a solution to the Nash condition, it is verified*:

$$\lim_{v\to\infty} f(v) > 0 \ .$$

Proof. Any solution to the Nash condition must be such that the resulting $q(v)$ of Eq. (2) is a well-defined probability. Then, we can use Appendix D where it is proven that we can write

$$f(v) = D' \mathrm{e}^{J(v)} \ , \tag{A.1}$$

with $D' > 0$ and

$$J(v) \equiv \frac{I_+(v)}{v} + \int_1^v \frac{I_+(z)}{z^2}\,\mathrm{d}z \ ,$$

being $I_+(v) \geqslant 0 \quad$ for all $v \geqslant 1$.

We have that $J(v) \geqslant 0$ for all $v \geqslant 1$ because it is the sum of two positive terms, thus $f(v) \geqslant D'$ for all $v \geqslant 1$ and, as a consequence, $\lim_{v\to\infty} f(v) > 0$ because $D' > 0$. □

Lemma A.2. *If $f(v)$ is an efficient solution to the Nash condition, it is verified*:

$$\lim_{v\to\infty} f(v) < \infty \, .$$

Proof. It is done by *reductio ad absurdum*.

Since any solution $f(v)$ must result in a well-defined probability function $q(v)$, it is proven in Appendix D that it takes the form given by Eq. (A.1). Thus if we suppose that $\lim_{v\to\infty} f(v) = \infty$, it is because $\lim_{v\to\infty} J(v) = \infty$.

If such a solution is also efficient, the following condition must hold:

$$0 < \lim_{v\to\infty} \lambda^*(v) q[\lambda(v)] < \infty \, ,$$

which in the continuum limit implies that

$$0 < \lim_{v\to\infty} f(v) \mathrm{e}^{-I_+(v)} < \infty \, .$$

Taking into account Eq. (A.1), the former condition can be written as

$$0 < \lim_{v\to\infty} \mathrm{e}^{J(v)} \mathrm{e}^{-I_+(v)} < \infty \, ,$$

which can only be true if

$$\lim_{v\to\infty} J(v) - I_+(v) \neq \pm\infty \, . \tag{A.2}$$

Since $\lim_{v\to\infty} J(v) = \infty$, Eq. (A.2) can only be satisfied if $\lim_{v\to\infty} I_+(v) = \infty$. But, if it is verified that $\lim_{v\to\infty} J(v) - I_+(v) = \text{constant}$, with $\lim_{v\to\infty} J(v) = \infty$ and $\lim_{v\to\infty} I_+(v) = \infty$, it is easy to check that

$$\lim_{v\to\infty} \frac{J(v)}{I_+(v)} = 1 \, ,$$

which implies, by definition of $J(v)$, that

$$\lim_{v\to\infty} \frac{1}{I_+(v)} \int_1^v \frac{I_+(z)}{z^2} \, \mathrm{d}z = 1 \, . \tag{A.3}$$

Let us define the function $s(v)$ as follows:

$$s(v) \equiv \frac{1}{I_+(v)} \int_1^v \frac{I_+(z)}{z^2} \, \mathrm{d}z \, .$$

We have that Eq. (A.3) imposes on $s(v)$ the condition $\lim_{v\to\infty} s(v) = 1$. Since it is also true that $\lim_{v\to\infty} 1/s(v) = 1$, there exists some $V \geqslant 1$ such that, for all $v > V$,

$$\frac{1}{s(v)} < 1 + \varepsilon \, , \tag{A.4}$$

where ε is a positive real number.

On the other hand, by definition of $s(v)$, its derivative with respect to v is the following:

$$\frac{I_+}{v^2} = \dot{s}(v)I_+ + s(v)\dot{I}_+ \,.$$

Then, the following ordinary differential equation is obtained:

$$\frac{\dot{I}_+}{I_+} = \frac{1}{s(v)v^2} - \frac{\dot{s}(v)}{s(v)} \,,$$

whose solution can be written formally as

$$I_+(v) = \frac{C}{s(v)} \exp\left[\int \frac{\mathrm{d}v}{s(v)v^2}\right] ,$$

where C is a constant of integration.

If Eq. (A.4) is considered, we have that, for all $v > V$,

$$I_+(v) < C(1+\varepsilon)\exp\left[\int \frac{1+\varepsilon}{v^2}\,\mathrm{d}v\right] = C(1+\varepsilon)\exp\left[C' - \frac{1+\varepsilon}{v}\right] ,$$

where C' is a constant of integration. As a consequence, it is derived that, for all $v > V$

$$I_+(v) < C(1+\varepsilon)\exp\left[C' - \frac{1+\varepsilon}{V}\right] = C'' \,,$$

where C'' is a constant.

Therefore, it is obtained that $\lim_{v\to\infty} I_+(v) < \infty$. However, this contradicts the result derived from Eq. (A.2) (i.e., $\lim_{v\to\infty} I_+(v) = \infty$). Then, we conclude that no efficient solution to the Nash condition can verify that $\lim_{v\to\infty} f(v) = \infty$. □

Appendix B

Lemma. *If $\tilde{f}(v)$ is a twice differentiable function which behaves asymptotically as*

$$\dot{\tilde{f}}(v)v \sim \frac{1}{v^{1+\alpha}} \quad \text{with } \alpha > 0 \,,$$

it is an efficient solution to the Nash condition.

Proof. By definition of $\tilde{f}(v)$, it is easy to check that

$$\lim_{v\to\infty} \frac{1}{1+\tilde{f}(v)} = 1 \,.$$

Thus, given any $\varepsilon > 0$, there exists some $V \geqslant 1$ such that, for all $v > V$,

$$\left|\frac{1}{1+\tilde{f}(v)}\right| < 1 + \varepsilon \,.$$

Then, it is deduced that

$$\left|\int_1^\infty \frac{\dot{\tilde{f}}(v)}{1+\tilde{f}(v)} v\,\mathrm{d}v\right| < \int_1^\infty \left|\frac{\dot{\tilde{f}}(v)}{1+\tilde{f}(v)} v\right| \mathrm{d}v < B + (1+\varepsilon)\int_V^\infty |\dot{\tilde{f}}(v)| v\,\mathrm{d}v\,,$$

where B is a real number defined as

$$\int_1^V \left|\frac{\dot{\tilde{f}}(v)}{1+\tilde{f}(v)} v\right| \mathrm{d}v\,.$$

If $\dot{\tilde{f}}(v)v \sim v^{-(1+\alpha)}$ with $\alpha > 0$, and taking into account that the integration of a function which asymptotically behaves as $v^{-(1+\alpha)}$ is a function which tends to zero as $v \to \infty$, it is straightforward to check that

$$\int_V^\infty |\dot{\tilde{f}}(v)| v\,\mathrm{d}v = B'\,,$$

with B' being a constant.

Therefore, it is deduced that

$$\left|\int_1^\infty \frac{\dot{\tilde{f}}(v)}{1+\tilde{f}(v)} v\,\mathrm{d}v\right| < B + (1+\varepsilon)B' < \infty\,,$$

which implies that there exists an efficient solution (see Appendix E). □

Appendix C

Lemma. *If the Nash condition is verified efficiently, the following equation holds*:

$$\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 = 0\,.$$

Proof. Assume that $\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 = A' > 0$. Then, for every $A'' \in (0, A')$, it is always possible to find a $V_1 \geqslant 1$ such that $\dot{\tilde{f}}(v)v^2 \geqslant A''$, for all $v > V_1$. On the other hand, $\lim_{t\to\infty} \tilde{f}(v) = 0$. Thus for every $\varepsilon \in (0,1)$ there exists a $V_2 \geqslant 1$ such that

$$\frac{1}{1+\tilde{f}} > 1 - \varepsilon$$

for all $v > V_2$.

Then, we deduce by means of Appendix E that, for all $v > V \equiv \max\{V_1, V_2\}$,

$$\begin{aligned}\text{cons} = \int_1^\infty \frac{\dot{\tilde{f}}(v)v^2}{1+\tilde{f}(v)} \frac{\mathrm{d}v}{v} &\geqslant A + A'' \int_V^\infty \frac{1}{1+\tilde{f}(v)} \frac{\mathrm{d}v}{v} \\ &\geqslant A + A''(1-\varepsilon)\int_V^\infty \frac{\mathrm{d}v}{v} = \infty\,,\end{aligned}$$

because A is a constant defined as

$$A \equiv \int_1^V \frac{\dot{\tilde{f}}(v)}{1+\tilde{f}(v)} v \, \mathrm{d}v \, .$$

This contradiction arises because it was supposed that $\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 > 0$.

Similarly, when $\lim_{t\to\infty} \dot{\tilde{f}}(v)v^2 = -A' < 0$, for every $A'' \in (0, A')$, there exists some $V_1 \geqslant 1$ such that $\dot{\tilde{f}}(v)v^2 \leqslant -A''$, for all $v > V_1$. In addition, $\lim_{v\to\infty} \tilde{f}(v) = 0$. Thus given any $\varepsilon > 0$ there exists some $V_2 \geqslant 1$ such that

$$\frac{1}{1+\tilde{f}} < 1 + \varepsilon$$

for all $v > V_2$.

Then, it is derived from Appendix E that, for all $v > V \equiv \max\{V_1, V_2\}$,

$$\begin{aligned} \text{cons} = \int_1^\infty \frac{\dot{\tilde{f}}(v)v^2}{1+\tilde{f}(v)} \frac{\mathrm{d}v}{v} &\leqslant A - A'' \int_V^\infty \frac{1}{1+\tilde{f}(v)} \frac{\mathrm{d}v}{v} \\ &\leqslant A - A''(1+\varepsilon) \int_V^\infty \frac{\mathrm{d}v}{v} = -\infty \, . \end{aligned}$$

Such a contradiction only disappears when the condition $\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 < 0$ is rejected.

Notice that it is only possible to find an $A'' > 0$ when A' is not zero. If A' is zero the former reasoning fails because the first case results in cons$< \infty$ and the second one derives in cons$> -\infty$ and nothing can be argued.

These results imply that, given an efficient solution to the Nash condition, it is not feasible that $\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 \neq 0$. But, it could be possible that the condition $\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 = 0$ was also rejected. In that case, it would derive that there is no efficient solution to the Nash condition.

However, in Appendix B we demonstrate that the functions $\tilde{f}(v)$ which behave asymptotically as $\dot{\tilde{f}}(v)v \sim v^{-(1+\alpha)}$, with $\alpha > 0$, are efficient solutions. But these functions verify that $\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 = 0$, thus this condition defines a non-empty set of efficient solutions. Therefore, it can be affirmed that $\lim_{v\to\infty} \dot{\tilde{f}}(v)v^2 = 0$ is a necessary condition to obtain an efficient solution to the Nash condition. □

Appendix D

Lemma. *The function $q(v)$ is well-defined probability if and only if the solution to the Nash condition can be written as*

$$f(v) = D' \mathrm{e}^{J(v)} \, , \tag{D.1}$$

with $D' > 0$ and

$$J(v) \equiv \frac{I_+(v)}{v} + \int_1^v \frac{I_+(z)}{z^2} \, \mathrm{d}z \, ,$$

being $I_+(v) \geqslant 0$ for all $v \geqslant 1$.

Proof. Assume that $q(v)$ is a probability which verifies the Nash condition for all $v \geqslant 1$. Then, from Eq. (2) we derive that $I(v) \neq -\infty$ for all $v \geqslant 1$ (otherwise, $q(v) > 1$ and it would not be a probability). Therefore, we can define a new function $I_+(v)$ such that $I(v) \equiv -M + I_+(v)$, with $0 \leqslant M < \infty$ and $I_+(v) \geqslant 0$ for all $v \geqslant 1$.

The definition of $I_+(v)$ implies that

$$\dot{I}_+(v) = \frac{\dot{f}(v)}{f(v)} v ,$$

which is an ordinary differential equation easily integrable in this manner

$$\ln \left[\frac{f(v)}{f(1)} \right] = \int_1^v \frac{\dot{I}_+(z)}{z} \, \mathrm{d}z = \frac{I_+(v)}{v} - I_+(1) + \int_1^v \frac{I_+(z)}{z^2} \, \mathrm{d}z ,$$

where the right-hand side of the equation is obtained from the integration by parts.

Therefore, if $q(v)$ is a well-defined probability for all $v \geqslant 1$, the function $f(v)$ can be written as

$$f(v) = D' \mathrm{e}^{J(v)} ,$$

where the function $J(v)$ is

$$J(v) \equiv \frac{I_+(v)}{v} + \int_1^v \frac{I_+(z)}{z^2} \, \mathrm{d}z ,$$

and $D' \equiv f(1) \mathrm{e}^{-I_+(1)}$ is a positive constant because $f(1) = \lambda^*(1) > 0$.

Now, assume that the function $f(v)$ is the one described by Eq. (D.1). It is easy to verify that

$$I(v) = \int \frac{\dot{f}(v)}{f(v)} v \, \mathrm{d}v = I_+(v) + D'' ,$$

where D'' is another constant of integration. Consequently, $q(v)$ can be written as

$$q(v) = D \mathrm{e}^{-D''} \mathrm{e}^{-I_+(v)} .$$

But we have that $I_+(v) \geqslant 0$ for all $v \geqslant 1$, which implies that $\mathrm{e}^{-I_+(v)} \leqslant 1$ for all $v \geqslant 1$. Likewise, the constant $D\mathrm{e}^{-D''}$ is an arbitrary constant since D is also arbitrary. Then, D can be defined to keep $q(v)$ in the interval $[0, 1]$ for all $v \geqslant 1$ and, therefore, a well-defined probability is obtained. □

Appendix E

Lemma. *There exists an efficient solution to the Nash condition if and only if*

$$\int_1^\infty \frac{\dot{\tilde{f}}(v)}{1 + \tilde{f}(v)} v \, \mathrm{d}v \neq \pm \infty . \tag{E.1}$$

Proof. The function f must be such that the result of Eq. (2) is a well-defined probability (i.e., in the range $[0, 1]$). Then, f is a solution to the Nash condition if and only if $I(v) \neq -\infty$ for all $v \geqslant 1$.

But from the mean-value theorem for integrals [19], it is derived that $I(v)$ is a constant for all $1 \leqslant v < \infty$. Note that it is the integration of a continuous function in a finite interval (recall that $f(v)$ is defined as twice differentiable). Hence, in order to obtain a solution to the Nash condition, it is not necessary to verify that $I(v) > -\infty$ for all $v \geqslant 1$ but to check that $\lim_{v\to\infty} I(v) \neq -\infty$.

On the other hand, since $\lim_{v\to\infty} f(v) = f_\infty$, the condition of efficiency is verified when $\lim_{v\to\infty} q(v) \neq 0$. Then, from Eq. (2) is deduced that the efficiency is guaranteed if and only if $\lim_{v\to\infty} I(v) \neq \infty$.

Therefore, the condition $\lim_{v\to\infty} I(v) \neq \pm\infty$ is satisfied if and only if $f(v)$ is an efficient solution. Thus writing $I(v)$ in terms of the $\tilde{f}(v)$ defined in Eq. (3), the following condition must hold:

$$\pm\infty \neq \lim_{v\to\infty} I(v) = \text{cons} + \lim_{v\to\infty} \int_1^v \frac{\dot{\tilde{f}}(z)}{1+\tilde{f}(z)} z \, \mathrm{d}z \,,$$

which is equivalent to Eq. (E.1). □

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